THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLOWS.

CONTENTS

NOMENCLATURE

e : specific total energy
$\underset{\sim}{f}$: force
F, G, H, U : vector fluxes
J : Jacobian of coordinate transformation
k : molecular heat conductivity
$\underset{\sim}{n}$: outward normal
p : pressure
Pr : Prandtl number
Pr_t : turbulent Prandtl number
q : heat exchange
Q : heat release
t : time
T : temperature
u, v, w : velocity components in Cartesian frame
x, y, z : Cartesian coordinates
γ : ratio of specific heats C_p / C_v
μ_t : eddy viscosity
μ : molecular viscosity coefficient
λ : bulk viscosity coefficient
ξ, η, ζ : transformed coordinates
ρ : density
$\underset{\approx}{\tau}$: viscous stress tensor

1. Introduction.

The three dimensional, time-dependent, compressible Navier-Stokes' equations are the cornerstone of Fluid dynamics. Applied to a control volume V enclosed by a control surface A, these equations express the conservation of mass, momentum and energy. However these equations in their integral form are not very usable. That is why, using the transport theorems, we transform them into differential equations. Moreover these equations are expressed in a coordinate system, generally the Cartesian coordinate system in which the derivation of these equations is much more simpler than in any other coordina system, and rewritten in a flux vector form which is the basic equation in the numerical procedures to solve the Navier-Stokes' equations. However, when studying the flow structure around a body such as an aircraft, the Cartesian coordinates are rarely adequate in describing the geometric configurations of the aircraft and consequently it becomes necessary to introduce a generalized coordinate mapping for each configuration and derive the strong conservation form of the Navier-Stokes' equations.

Furthermore, in the study of flows around a body such as an aircraft, experiment has shown that it is not always necessary to solve the complete three dimensional, time-depende compressible Navier-Stokes' equations. The resolution of approximate Navier-Stokes' equations often gives very satisfactory results and enables to save considerable computationa time. Depending on the zone of study, it is possible to apply some or other approximate Navier-Stokes' equation. The zonal method consists in solving these approximate Navier-Stokes' equations and then obtaining a composite solution for the overall flow structure. If generally the zonal method is particularly efficient and accurate, however in the situation where multiple interacting zones are tightly knitted together or an unsteady phenomenon arises, the advantage of the zonal approach over the full equation solution is uncertain. This will not be discussed in the present paper but let us remember that the zonal method is a vast subject of research. In the range between the complete Navier-Stokes' equations and the boundary layer equations, the approximate Navier-Stokes' equations most used for compressible flows are;

- The "thin-layer" equations
- The "parabolized" Navier-Stokes' equations
- The "conical" Navier-Stokes' equations
- The Viscous Shock-Layer equations

There are many numerical solutions to the compressible Navier-Stokes' Equations. However in the present paper, only the finite-difference procedures are addressed. Among them, the most frequently used are the MacCormack's algorithms, the Beam and Warming's approximate factorization scheme, the Briley and MacDonald's procedure and several hybrid methods. In this present effort, the scheme of an explicit method and the scheme of an implicit method are given: the Explicit MacCormack's method and the Beam and Warming's scheme.

The principal aim of this paper is to introduce the reader to the numerical solutions of the compressible Navier-Stokes' equations:

- What is the general form of the equations we wish to solve?
- How and for which configurations these equations can be simplified?
- Which finite-difference schemes are applicable to these equations?

2. Derivation of the compressible Navier-Stokes' Equations.

In the Eulerian formulation, the conservative equations in integral form for mass, momentum and energy with respect to a control volume V <u>stationary</u> in the initial frame and enclosed by the control surface A are:

- (1) $\frac{\partial}{\partial t}\iiint_V \rho \, dV + \iint_A \rho \, \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA = 0$ or $\frac{\partial}{\partial t}\iiint_V \rho \, dV + \iint_A \rho \, u_i n_i \, dA = 0$

- (2) $\frac{\partial}{\partial t}\iiint_V \rho \, \underset{\sim}{u} \, dV + \iint_A \rho(\underset{\sim}{u}\,\underset{\sim}{u}) \, \underset{\sim}{n} \, dA = \iint_A [-p \underset{\approx}{\delta} + \underset{\approx}{\tau}] \, \underset{\sim}{n} \, dA + \iiint_V \rho \underset{\sim}{f} \, dV$

  or $\frac{\partial}{\partial t}\iiint_V \rho \, u_i \, dV + \iint_A \rho \, u_i u_j n_j \, dA = \iint_A [-p \delta_{ij} + \tau_{ij}] n_j \, dA + \iiint_V \rho f_i \, dV$ (i=1,:

- (3) $\frac{\partial}{\partial t}\iiint \rho e \, dV + \iint_A \rho e \, \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA = \iint_A \left([-p \underset{\approx}{\delta} + \underset{\approx}{\tau}] \underset{\sim}{u} - q\right) \cdot \underset{\sim}{n} \, dA + \iiint_V \rho (\underset{\sim}{f} \cdot \underset{\sim}{u} + \dot{Q}) \, d$

  or $\frac{\partial}{\partial t}\iiint \rho e \, dV + \iint_A \rho e \, u_i n_i \, dA = \iint_A \left([-p \delta_{ij} + \tau_{ij}] u_j - q_i\right) n_i \, dA + \iiint_V \rho (f_i u_i + \dot{Q}) \, d$

<u>Notes</u>

- The first equation sets that matter can be neither created nor destroyed.
- The second equation is the Newton's second law of motion:

$$\left\{\begin{matrix}\text{acceleration}\\ \text{quantity}\end{matrix}\right\} = \left\{\begin{matrix}\text{pressure} & + & \text{viscous}\\ \text{forces} & & \text{forces}\end{matrix}\right\} + \left\{\begin{matrix}\text{body}\\ \text{forces}\end{matrix}\right\}$$

- The principle of conservation of energy (equation (3)) states that:

$$\left\{\begin{matrix}\text{rate of increase}\\ \text{of total energy in V}\end{matrix}\right\} = \left\{\begin{matrix}\text{flux of energy}\\ \text{entering V accross A}\end{matrix}\right\} + \left\{\begin{matrix}\text{work per unit of time}\\ \text{of the pressure and viscous forces}\end{matrix}\right\} + \left\{\begin{matrix}\text{heat flux}\\ \text{entering V accross A}\end{matrix}\right\}$$

$$+ \left\{\begin{matrix}\text{work per unit of time}\\ \text{of the body forces}\end{matrix}\right\} + \left\{\begin{matrix}\text{energy generated}\\ \text{inside the volume V}\end{matrix}\right\}$$

Now, since the control volume is stationary, the relative order of time derivativ and volume integral may be exchanged, and by means of the divergence theorem we can derive the corresponding differential equation:

- (4) $\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \underset{\sim}{u}) = 0$ or $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$

- (5) $\frac{\partial}{\partial t}(\rho \underset{\sim}{u}) + \nabla \cdot \left(\rho(\underset{\sim}{u}\,\underset{\sim}{u}) + p \underset{\approx}{\delta} - \underset{\approx}{\tau}\right) - \rho \underset{\sim}{f} = \underset{\sim}{0}$

  or $\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p \delta_{ij} - \tau_{ij}\right) - \rho f_i = 0$ (i=1,2,3)

(6) $$\frac{\partial}{\partial t}(\rho e) + \nabla\cdot\left(\rho e \underset{\sim}{u} + p \underset{\approx}{\delta}\, \underset{\sim}{u} - \underset{\approx}{\tau}\, \underset{\sim}{u} + \underset{\sim}{q}\right) - \rho\left(\underset{\sim}{f}\, \underset{\sim}{u} + \dot{Q}\right) = 0$$

or $$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}\left(\rho e\, u_i + p\, u_i - \tau_{ij} u_j + q_i\right) - \rho\left(f_i u_i + \dot{Q}\right) = 0$$

or $$\frac{\partial}{\partial t}(\rho C_p T) - \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho C_p T u_i - \tau_{ij} u_j + q_i\right) - \rho\left(f_i u_i + \dot{Q}\right) = 0$$

In these equations, the viscous stress tensor is given as:

(7) $$\underset{\approx}{\tau} = \lambda\, \nabla\cdot\underset{\sim}{u}\ \underset{\approx}{\delta} + \mu\left(\nabla\underset{\sim}{u} + \nabla\underset{\sim}{u}^T\right)$$

or $$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

And the heat flux vector is given by the Fourier's law for heat transfer:

(8) $$\underset{\sim}{q} = -k\,\nabla T \qquad \text{or} \qquad q_i = -k\frac{\partial T}{\partial x_i}$$

Let us change now our notation in the following way:

$$\underset{\sim}{u} \longrightarrow (u, v, w) \qquad\qquad (x_1, x_2, x_3) \longrightarrow (x, y, z)$$

And let us introduce equations (7) and (8) in the equations (4), (5_1), (5_2), (and (6). We get a system of five equations where there are ten unknowns ($u, v, w, \rho, p, T, e, k, \mu, \lambda$). Therefore additional equations must be included.

For most aerodynamics applications we add the following equations:

(9) The perfect gas equation of state: $\frac{p}{\rho} = RT$

(10) The calorically perfect gas assumptions:

$$e = C_v T \qquad h = C_p T \qquad \gamma = \frac{C_p}{C_v} \qquad C_v = \frac{R}{\gamma - 1} \qquad C_p = \frac{\gamma R}{\gamma - 1}$$

(11) The Sutherland's viscosity law:

$$\mu = C_1 \frac{T^{3/2}}{T + C_2}$$

(12) The Prandtl number is assumed to be constant: $Pr = \frac{C_p \mu}{k}$

(13) The Stoke's hypothesis (zero bulk viscosity) is assumed to be satisfied:

$$3\lambda + 2\mu = 0$$

These five additional equations complete the description of the system of equations

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We have $\underset{\sim}{q} = -k \nabla T$ and $T = \frac{e}{C_v} = \frac{\gamma}{C_p} e$

Then $\underset{\sim}{q} = -\frac{k}{C_p} \gamma \nabla e = -\frac{\mu}{Pr} \gamma \nabla e$

- In the equation of energy expressed in terms of the partial derivatives of e we can replace $\underset{\sim}{q}$ by $-\frac{\mu}{Pr} \gamma \nabla e$

At this step of our discussion, we neglect now the body forces $\underset{\sim}{f}$ and the external heat addition $\dot{Q}$.

We must also consider separately the case when the flow is laminar and the case when the flow is turbulent.

For laminar flow, with appropriate initial and boundary conditions, the compressible Navier-Stokes' Equations in principle are solvable.

For turbulent flow, due to the limitation of computing facilities, we must transform the system of differential equations by time-averaging it and expressing it in terms of the mass averaged variables. The new set of variables is $(\tilde{u}, \tilde{v}, \tilde{w}, \bar{\rho}, \bar{p}, \tilde{T}, \tilde{e}, k, \mu, \lambda)$ where the mass averaged variables are:

$$\tilde{u} = \frac{\overline{\rho u}}{\bar{\rho}} \qquad \tilde{v} = \frac{\overline{\rho v}}{\bar{\rho}} \qquad \tilde{w} = \frac{\overline{\rho w}}{\bar{\rho}} \qquad \tilde{T} = \frac{\overline{\rho T}}{\bar{\rho}} \qquad \tilde{e} = \frac{\overline{\rho e}}{\bar{\rho}}$$

We thus derive the Reynolds-averaged Navier-Stokes' Equations. These equations are in general identical with the laminar flow counterparts provided that we add the apparent turbulent stresses in the stress tensor and the apparent turbulent heat flux vector in the heat flux vector.

Then, following Boussinesq approach, we can change the Navier-Stokes' Equations to a modeled form of the Reynolds-averaged equations by replacing the coefficie of viscosity μ with $\mu + \mu_T$ and the coefficient of thermal conductivity k with $k + k_T$ where:

μ_T is the eddy viscosity : $-\frac{2}{3}\mu_T (\nabla \cdot \underset{\sim}{\bar{u}}) \underset{\approx}{\delta} + \mu_T (\nabla \underset{\sim}{\bar{u}} + \nabla \underset{\sim}{\bar{u}}^T) = \underset{\approx}{\bar{\tau}}_{turb}$.

k_T is the turbulent thermal conductivité : $-k_T \nabla T = \underset{\sim}{q}_{turb}$

Then we can define the turbulent Prandtl number by $\frac{k_T}{C_p} = \frac{\mu_T}{Pr_T}$

And consequently: $\underset{\sim}{\tilde{q}} = -\frac{k + k_T}{C_p} \gamma \nabla \tilde{e} = -\gamma \left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right) \nabla \tilde{e}$

- In the equation of energy expressed in terms of the partial derivatives of $\tilde{e}$, we can replace $\underset{\sim}{\tilde{q}}$ by $-\gamma \left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right) \nabla \tilde{e}$

The Navier-Stokes' Equations for laminar flow and Reynolds-averaged Navier-Stokes' Equations are most frequently rewritten in flux vector form for Cartesian coordinates. This form is:

$$\boxed{(14)\qquad \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0}$$

where <u>for laminar flow</u> :

(15) $$U = (\rho\,,\ \rho u\,,\ \rho v\,,\ \rho w\,,\ \rho e\,)$$

(16) $$F = \begin{cases} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho uv - \tau_{xy} \\ \rho uw - \tau_{xz} \\ (\rho e + p)u - \gamma\left(\frac{\mu}{Pr}\right)\frac{\partial e}{\partial x} - (u\,\tau_{xx} + v\,\tau_{xy} + w\,\tau_{xz}) \end{cases}$$

(17) $$G = \begin{cases} \rho v \\ \rho uv - \tau_{xy} \\ \rho v^2 + p - \tau_{yy} \\ \rho vw - \tau_{yz} \\ (\rho e + p)v - \gamma\left(\frac{\mu}{Pr}\right)\frac{\partial e}{\partial y} - (u\,\tau_{xy} + v\,\tau_{yy} + w\,\tau_{yz}) \end{cases}$$

(18) $$H = \begin{cases} \rho w \\ \rho uw - \tau_{xz} \\ \rho vw - \tau_{yz} \\ \rho w^2 + p - \tau_{zz} \\ (\rho e + p)w - \gamma\left(\frac{\mu}{Pr}\right)\frac{\partial e}{\partial z} - (u\,\tau_{xz} + v\,\tau_{yz} + w\,\tau_{zz}) \end{cases}$$

and <u>for turbulent flow</u> : $\quad \bar q = \bar q_{lam} + \bar q_{turb} \quad ; \quad \bar\tau = \bar\tau_{lam.} + \bar\tau_{turb.}$

(15 bis) $$U = (\bar\rho\,,\ \bar\rho\tilde u\,,\ \bar\rho\tilde v\,,\ \bar\rho\tilde w\,,\ \bar\rho\tilde e\,)$$

(16 bis) $$F = \begin{cases} \bar\rho\,\tilde u \\ \bar\rho\,\tilde u^2 + \bar p - \bar\tau_{xx} \\ \bar\rho\,\tilde u\tilde v - \bar\tau_{xy} \\ \bar\rho\,\tilde u\tilde w - \bar\tau_{xz} \\ (\bar\rho\tilde e + \bar p)\tilde u - \gamma\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial\tilde e}{\partial x} - (\tilde u\,\bar\tau_{xx} + \tilde v\,\bar\tau_{xy} + \tilde w\,\bar\tau_{xz}) \end{cases}$$

(17 bis) $$G = \begin{cases} \bar\rho\,\tilde v \\ \bar\rho\,\tilde u\tilde v - \bar\tau_{xy} \\ \bar\rho\,\tilde v^2 + \bar p - \bar\tau_{yy} \\ \bar\rho\,\tilde v\tilde w - \bar\tau_{yz} \\ (\bar\rho\tilde e + \bar p)\tilde v - \gamma\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial\tilde e}{\partial y} - (\tilde u\,\bar\tau_{xy} + \tilde v\,\bar\tau_{yy} + \tilde w\,\bar\tau_{yz}) \end{cases}$$

(18 bis) $$H = \begin{cases} \bar\rho\,\tilde w \\ \bar\rho\,\tilde u\tilde w - \bar\tau_{xz} \\ \bar\rho\,\tilde v\tilde w - \bar\tau_{yz} \\ \bar\rho\,\tilde w^2 + \bar p - \bar\tau_{zz} \\ (\bar\rho\tilde e + \bar p)\tilde w - \gamma\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial\tilde e}{\partial z} - (\tilde u\,\bar\tau_{xz} + \tilde v\,\bar\tau_{yz} + \tilde w\,\bar\tau_{zz}) \end{cases}$$

These equations are derived with respect to the Cartesian coordinates. But for most engineering applications, this system of coordinates is rarely adequate in describing the geometric configuration. First because an extensive interpolation procedure for boundary conditions becomes necessary and second because it is difficult to implement a systematic clustering of grid spacing. Thus a generalized coordinate mapping is introduced in the form:

$$x = x(\xi, \eta, \zeta) \qquad y = y(\xi, \eta, \zeta) \qquad z = z(\xi, \eta, \zeta)$$

Then equation (14) becomes:

$$\frac{\partial U}{\partial t} + (\xi_x, \xi_y, \xi_z)\begin{pmatrix}\partial F/\partial\xi \\ \partial G/\partial\xi \\ \partial H/\partial\xi\end{pmatrix} + (\eta_x, \eta_y, \eta_z)\begin{pmatrix}\partial F/\partial\eta \\ \partial G/\partial\eta \\ \partial H/\partial\eta\end{pmatrix} + (\zeta_x, \zeta_y, \zeta_z)\begin{pmatrix}\partial F/\partial\zeta \\ \partial G/\partial\zeta \\ \partial H/\partial\zeta\end{pmatrix} = 0$$

Then,

$$\begin{pmatrix}d\xi \\ d\eta \\ d\zeta\end{pmatrix} = \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}^T \begin{pmatrix}dx \\ dy \\ dz\end{pmatrix} = \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}^T \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}^T \begin{pmatrix}d\xi \\ d\eta \\ d\zeta\end{pmatrix}$$

$$\Rightarrow \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)} = \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}^{-1} = J \; co\left(\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right)^T$$

where $J = \det\left(\frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)}\right) = 1/\det\left(\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right)$ is the Jacobian of the coordinate transfor

and $co\left(\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\right)^T$ is the transpose of the co-matrix of the matrix $\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}$

Thus

$$\begin{aligned}
\xi_x/J &= y_\eta z_\zeta - y_\zeta z_\eta \; ; & \eta_x/J &= y_\zeta z_\xi - y_\xi z_\zeta \; ; & \zeta_x/J &= y_\xi z_\eta - y_\eta z_\xi \\
\xi_y/J &= x_\zeta z_\eta - x_\eta z_\zeta \; ; & \eta_y/J &= x_\xi z_\zeta - x_\zeta z_\xi \; ; & \zeta_y/J &= x_\eta z_\xi - x_\xi z_\eta \\
\xi_z/J &= x_\eta y_\zeta - x_\zeta y_\eta \; ; & \eta_z/J &= x_\zeta y_\xi - x_\xi y_\zeta \; ; & \zeta_z/J &= x_\xi y_\eta - x_\eta y_\xi
\end{aligned}$$

Then let us divide the transformed equation by the Jacobian and rearrange by adding and subtracting like terms. We have:

$$\left(\frac{U}{J}\right)_t + \left(\frac{F\xi_x + G\xi_y + H\xi_z}{J}\right)_\xi + \left(\frac{F\eta_x + G\eta_y + H\eta_z}{J}\right)_\eta + \left(\frac{F\zeta_x + G\zeta_y + H\zeta_z}{J}\right)_\zeta$$

$$- F\left[\left(\frac{\xi_x}{J}\right)_\xi + \left(\frac{\eta_x}{J}\right)_\eta + \left(\frac{\zeta_x}{J}\right)_\zeta\right] - G\left[\left(\frac{\xi_y}{J}\right)_\xi + \left(\frac{\eta_y}{J}\right)_\eta + \left(\frac{\zeta_y}{J}\right)_\zeta\right] - H\left[\left(\frac{\xi_z}{J}\right)_\xi + \left(\frac{\eta_z}{J}\right)_\eta + \left(\frac{\zeta_z}{J}\right)_\zeta\right] =$$

The last three terms are all equal to zero and can be dropped. And thus, the equations in the new system of coordinates take <u>the strong conservation form</u>:

$$\boxed{(19) \quad \frac{\partial}{\partial t}(U/J) + \frac{\partial}{\partial\xi}\left[(F\xi_x + G\xi_y + H\xi_z)/J\right] + \frac{\partial}{\partial\eta}\left[(F\eta_x + G\eta_y + H\eta_z)/J\right] + \frac{\partial}{\partial\zeta}\left[(F\zeta_x + G\zeta_y + H\zeta_z)/J\right] = 0}$$

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Thus we derived the strong conservation form for the compressible (laminar or turbulent) Navier-Stokes' Equations. In these equations the right hand side term is zero because we neglected the body forces $\underset{\sim}{f}$ and the external heat addition $\dot{Q}$. If we had not neglected these terms, they would have appeared in the right hand side of these equations.

Note: • The partial derivative terms appearing in U, F, G and H are to be transformed too. For example the viscous stress tensor would be transformed to:

$$\tau_{ii} = \frac{4\mu}{3}\left(\xi_i \frac{\partial u_i}{\partial \xi} + \eta_i \frac{\partial u_i}{\partial \eta} + \zeta_i \frac{\partial u_i}{\partial \zeta}\right) - \frac{2\mu}{3}\left(\xi_j \frac{\partial u_j}{\partial \xi} + \eta_j \frac{\partial u_j}{\partial \eta} + \zeta_j \frac{\partial u_j}{\partial \zeta} + \xi_k \frac{\partial u_k}{\partial \xi} + \eta_k \frac{\partial u_k}{\partial \eta} + \zeta_k \frac{\partial u_k}{\partial \zeta}\right)$$

$(i, j, k) =$ any permutation of (x, y, z)

$$\tau_{ij} = \mu\left(\xi_j \frac{\partial u_i}{\partial \xi} + \eta_j \frac{\partial u_i}{\partial \eta} + \zeta_j \frac{\partial u_i}{\partial \zeta} + \xi_i \frac{\partial u_j}{\partial \xi} + \eta_i \frac{\partial u_j}{\partial \eta} + \zeta_i \frac{\partial u_j}{\partial \zeta}\right)$$

$(i, j) = (x, y), (x, z)$ or (y, z)

• If more generally, we had assumed a coordinate transformation in the form:

$$x = x(\xi, \eta, \zeta, t) \qquad y = y(\xi, \eta, \zeta, t) \qquad z = z(\xi, \eta, \zeta, t)$$

Then the strong conservation form would have taken the form:

$$\frac{\partial}{\partial t}(U/J) + \frac{\partial}{\partial \xi}\left[(U\xi_t + F\xi_x + G\xi_y + H\xi_z)/J\right] + \frac{\partial}{\partial \eta}\left[(U\eta_t + F\eta_x + G\eta_y + H\eta_z)/J\right]$$

$$+ \frac{\partial}{\partial \zeta}\left[(U\zeta_t + F\zeta_x + G\zeta_y + H\zeta_z)/J\right] = 0 .$$

3. Approximate Navier - Stokes' Equations.

It is very difficult to solve the complete Navier - Stokes' Equations, but fortunately, for many studies of flows around a body where the boundary-layer equations are not applicable, it is however possible to solve a reduced set of equations that fall between the complete Navier - Stokes' equations and the boundary layer equations in terms of complexity. most frequently used approximate Navier - Stokes' Equations are:

- The "thin - layer" Navier - Stokes' equations
- The "parabolized" Navier - Stokes' equations
- The "conical" Navier - Stokes' equations
- The Viscous-shock - layer equations

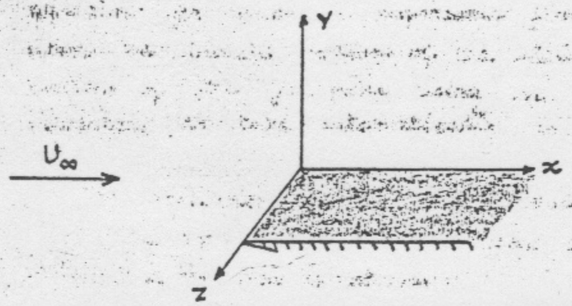
3.1. The "thin - layer" Navier - Stokes' equations.

The principle is to neglect all the viscous terms containing derivatives in the directions parallel to the body surface in the unsteady Navier - Stokes' equations but to retain all other terms.

There are two principal advantages of retaining the terms which are normally neglected in boundary - layer theory:

- separated and reverse flow regions can be directly computed since adverse pressure gradients are taken in account.
- flows which contain a large normal pressure gradient can be easily computed.

Example: for a laminar flow over a flat plate;



$$\mu \frac{\partial \Theta}{\partial x} = \mu \frac{\partial \Theta}{\partial z} \cong 0$$

where

$$\Theta = \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}$$

(20)

continuity: $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

x momentum: $$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(p + \rho u^2) + \frac{\partial}{\partial y}\left(\rho u v - \mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}(\rho u w) = 0$$

y momentum: $$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}\left(p + \rho v^2 - \frac{4}{3}\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}(\rho v w) = 0$$

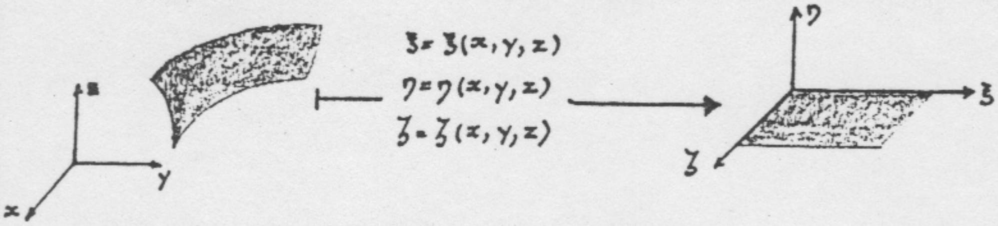
z momentum: $$\frac{\partial \rho w}{\partial t} + \frac{\partial}{\partial x}(\rho u w) + \frac{\partial}{\partial y}\left(\rho v w - \mu \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}(p + \rho w^2) = 0$$

energy: $$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho e u + p u) + \frac{\partial}{\partial y}\left(\rho e v + p v - \mu u \frac{\partial u}{\partial y} - \frac{4}{3}\mu v \frac{\partial v}{\partial y} - \mu w \frac{\partial w}{\partial y} - \frac{\mu}{P_r}\gamma \frac{\partial e}{\partial y}\right) + \frac{\partial}{\partial z}(\rho e w + p w) = 0$$

Let us note that viscous terms appear as well in the viscous stress tensor as in the heat flux since $q = -\frac{\mu}{Pr}\gamma \nabla e$. Therefore we retain only the heat flux in the y - direction: $- \mu \gamma \frac{\partial e}{}$

For more complicated body geometries, we do:

- a mapping of the body surface into a transformed coordinate surface.



- use the strong-conservation form with the flux vectors given by the preceding formulation.

In conclusion, the "thin-layer" Navier-Stokes' Equations are much less complicated than the complete Navier-Stokes' Equations but still require a large amount of computer effort to be solved. They are a mixed set of hyperbolic-parabolic partial differential equations in time and consequently can be solved by the time-dependent procedure

3.2. The "parabolized" Navier-Stokes' Equations. (PNS)

These equations gained popularity because they can be very efficiently used to predict complex three-dimensional, steady, supersonic viscous flowfields. Solving the PNS equations for an entire supersonic flowfield does not require more effort than solving either the inviscid portion of the flowfield using the Euler equations or the viscous portion of the flowfield using the boundary-layer equations and has the great advantage to take automatically in account the interaction between these regions.

The derivation of the PNS equations assume certain conditions to be met:
- the inviscid outer region of the flow is supersonic
- the flow is steady
- the streamwise velocity component is everywhere positive, i.e. streamwise flow separation is excluded

Rudman and Rubin (1968) were among the firsts to derive the PNS equations and use them for the study of the hypersonic laminar flow near the leading edge of a flat plate. Their process was the following:

(i) nondimensionalize the flow variables with respect to the freestream values and the space coordinates with respect to characteristic length (L in the x-direction, δ in the y-direction):

$$u = u^* V_\infty \qquad v = v^* V_\infty \qquad p = p^* p_\infty \qquad \rho = \rho^* \rho_\infty \qquad T = T^* T_\infty \qquad \mu = \mu^* \mu_\infty$$

$$x = x^* L \qquad y = y^* \qquad \text{and} \qquad \delta = \delta^* L$$

(ii) assume series expansions of the nondimensionalize flow variables with respect to local flow variables nondimensionalized with respect to local reference conditions:

$$u^* = u_0^* + \epsilon u_1^* + \cdots$$
$$v^* = \delta^* (v_0^* + \epsilon v_1^* + \cdots)$$

$$p^* = p^*_{ref}\,(p_0^* + \epsilon\, p_1^* + \cdots)$$
$$\rho^* = \rho^*_{ref}\,(\rho_0^* + \epsilon\, \rho_1^* + \cdots)$$
$$T^* = T^*_{ref}\,(T_0^* + \epsilon\, T_1^* + \cdots)$$
$$\mu^* = \mu^*_{ref}\,(\mu_0^* + \epsilon\, \mu_1^* + \cdots)$$

The zeroth-order solution is obtained from the first term in the series expansions while the first-order solution is obtained from both the first and second term in the series expansions.

Furthermore in the relatively thin region where the flow is disturbed, the gradients normal to the surface are much greater than the gradients parallel to the surface and δ^* can be assumed to be small.

(iii) substitute the expansions into the two-dimensional, steady, Navier-Stokes equations and do a magnitude analysis.

The result is that:

– The zeroth-order equations are derived by neglecting terms of order $(\delta^*)^2$, $\Delta^2 = T^*_{ref} / M_\infty^2 \gamma$ and ϵ. According to Rudman and Rubin, in order for $(\delta^*)^2$ to be very small (i.e. ≤ 0.05) the zeroth-order equations are <u>not valid upstream of the point at which:</u>

$$\frac{\chi_\infty}{M_\infty^2} \simeq 2 \qquad \text{where} \qquad \chi_\infty = \left(\frac{\mu_w T_\infty}{\mu_\infty T_w}\right)^{1/2} \left(M_\infty^3\, Re_{x\infty}\right)^{-1/2}$$

The zeroth-order equations are also applicable only when <u>$M_\infty \geq 5$</u> :

continuity:
$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

x-momentum:
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y-momentum:
$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{4}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial x}\right)$$

energy:
$$\rho u c_v \frac{\partial T}{\partial x} + \rho u c_v \frac{\partial T}{\partial y} = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{4}{3}\mu \left(\frac{\partial v}{\partial y}\right)^2$$

– The first-order equations are applicable when <u>$M_\infty \geq 2$</u>

<u>The three-dimensional equations</u> are derived by the same way. The coordinates x, y, z are nondimensionalized by L, δ_y and δ_z respectively. The velocities u, v, w are nondimensionalized by V_∞, $V_\infty \delta_y^*$ and $V_\infty \delta_z^*$ respectively where $\delta_y^* = \delta_y / L$ and $\delta_z^* = \delta_z / L$.

<u>The zeroth-order equations</u> become:

(21)

$$\text{continuity:}\quad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

$$\text{x-momentum:}\quad \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

$$\text{y-momentum:}\quad \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{4}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial x} + \mu \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial y}\right)$$

$$\text{z-momentum:}\quad \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{4}{3}\frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial y} + \mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial z}\right)$$

$$\text{energy:}\quad \rho u c_v \frac{\partial T}{\partial x} + \rho v c_v \frac{\partial T}{\partial y} + \rho w c_v \frac{\partial T}{\partial z} = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2\right] + \frac{4}{3}\mu\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right]$$

Let us note that in the PNS equations derived by Rudman and Rubin, there is no streamwise pressure gradient term and consequently the equations behave in a strictly "parabolic" manner in the boundary-layer region.

Another form of the PNS equations, including a streamwise pressure gradient term $-\partial p/\partial x$, and probably the most common form of the PNS equations is obtained by neglecting the streamwise viscous derivative terms, including heat flux terms, in $O(1)$ compare to the normal and transverse viscous derivative terms in $O(Re_L^{1/2})$. The equations in a Cartesian coordinate system (x is the streamwise direction) are:

(22)

continuity : same as equation (21)

$$\text{x-momentum:}\quad \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

$$\text{y-momentum:}\quad \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{4}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial y}\right) - \frac{2}{3}\frac{\partial}{\partial y}\left(\mu \frac{\partial}{\partial} \right.$$

$$\text{z-momentum:}\quad \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{4}{3}\frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial z}\right) - \frac{2}{3}\frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial y} \right.$$

energy : same as equation (21)

In order to apply the above equations to any more complex body surface, we must now consider the PNS equations in their strong conservation form, i.e., in terms of a generalized coordinate system. Recalling that the flow is steady, we can write:

$$\frac{\partial}{\partial \xi}\left[\left((F_i - F_v)\xi_x + (G_i - G_v)\xi_y + (H_i - H_v)\xi_z\right)/J\right] + \frac{\partial}{\partial \eta}\left[\left((F_i - F_v)\eta_x + (G_i - G_v)\eta_y + (H_i - H_v)\eta_z\right)/\right.$$

$$+ \frac{\partial}{\partial \zeta}\left[\left((F_i - F_v)\zeta_x + (G_i - G_v)\zeta_y + (H_i - H_v)\zeta_z\right)/J\right] = 0$$

where the flux vectors F, G and H have been decomposed in their inviscid and viscous componer

Then neglecting the streamwise viscous derivative terms, the strong conservation form for the PNS equations becomes: 12

(23)

$$\boxed{\begin{aligned}
&\frac{\partial \mathcal{F}}{\partial \xi} + \frac{\partial \mathcal{G}}{\partial \eta} + \frac{\partial \mathcal{H}}{\partial \zeta} = 0 \\
&\text{where} \quad \mathcal{F} = \frac{1}{J}\left(F_i \xi_x + G_i \xi_y + H_i \xi_z\right) \qquad (\xi = \text{streamwise direction}) \\
&\mathcal{G} = \frac{1}{J}\left((F_i - F_v')\eta_x + (G_i - G_v')\eta_y + (H_i - H_v')\eta_z\right) \\
&\mathcal{H} = \frac{1}{J}\left((F_i - F_v')\zeta_x + (G_i - G_v')\zeta_y + (H_i - H_v')\zeta_z\right)
\end{aligned}}$$

and F_v', G_v' and H_v' are the viscous components of the flux vectors F, G and H in which the terms containing partial derivatives with respect to ξ have been omitted.

For example τ_{xx}, τ_{xy}, τ_{xz} and q_x appear in F_v and

$$\tau_{xx} = \frac{2}{3}\mu\left[2(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta) - (\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta) - (\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta)\right]$$

$$\tau_{xy} = \mu(\xi_y u_\xi + \eta_y u_\eta + \zeta_y u_\zeta + \xi_x v_\xi + \eta_x v_\eta + \zeta_x v_\zeta)$$

$$\tau_{xz} = \mu(\xi_z u_\xi + \eta_z u_\eta + \zeta_z u_\zeta + \xi_x w_\xi + \eta_x w_\eta + \zeta_x w_\zeta)$$

$$q_x = -k(\xi_x T_\xi + \eta_x T_\eta + \zeta_x T_\zeta)$$

These terms are changed in τ_{xx}', τ_{xy}', τ_{xz}' and q_x' in F_v' where:

$$\tau_{xx}' = \frac{2}{3}\mu\left[2(\eta_x u_\eta + \zeta_x u_\zeta) - (\eta_y v_\eta + \zeta_y v_\zeta) - (\eta_z w_\eta + \zeta_z w_\zeta)\right]$$

$$\tau_{xy}' = \mu(\eta_y u_\eta + \zeta_y u_\zeta + \eta_x v_\eta + \zeta_x v_\zeta)$$

$$\tau_{xz}' = \mu(\eta_z u_\eta + \zeta_z u_\zeta + \eta_x w_\eta + \zeta_x w_\zeta)$$

$$q_x' = -k(\eta_x T_\eta + \zeta_x T_\zeta)$$

We have:

$$F_i = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (\rho e + p)u \end{pmatrix} \qquad G_i = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho u w \\ (\rho e + p)v \end{pmatrix} \qquad H_i = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (\rho e + p)w \end{pmatrix}$$

$$F_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{pmatrix} \qquad G_v = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{xy} + v\tau_{yy} + w\tau_{yz} - q_y \end{pmatrix} \qquad H_v = \begin{pmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ u\tau_{xz} + v\tau_{yz} + w\tau_{zz} - \end{pmatrix}$$

This strong conservation form in terms of the inviscid and viscous parts of the flux vectors is employed in numerical procedures for solving the PNS equations.

In many studies, the "thin-layer approximation" can also be applied to the PNS equations. The resulting equations are the steady form of the thin-layer Navier-Stokes' equations.

3.3. The "conical" Navier-Stokes' Equations.

A conical flow is such that all flow properties along the radial direction are constant and therefore a conical flow problem is a two-dimensional problem. But it is obvious that the above property implies that the flow be inviscid and thus the concept of conical flow is strictly valid only for inviscid flows. However experiments proved that the viscous portion of a flowfield was strongly dominated by the outer inviscid conical flow. Consequently, according to Anderson (1973), one can quickly estimate the heat transfer and skin friction for such flowfields by solving the unsteady Navier-Stokes' Equations in a time-dependent procedure on a sphere with all derivatives in the radial direction equal to zero.

Many flows around conical or pointed nose configurations have been computed using the conical approximation. In many cases the results provide starting solutions for PNS calculations.

The "conical" Navier-Stokes' equations are derived from the complete nondimensionalized Navier-Stokes equations:

$$\frac{\partial U^*}{\partial t} + \frac{\partial F^*}{\partial x^*} + \frac{\partial G^*}{\partial y^*} + \frac{\partial H^*}{\partial z^*} = 0$$

where the flux vectors U^*, F^*, G^* and H^* have exactly the same forms as those given in equations (15) to (18) by replacing the flow variables by their nondimensionalized counterparts which are:

$$u^* = u/V_\infty \quad v^* = v/V_\infty \quad w^* = w/V_\infty \quad \mu^* = \mu/\mu_\infty \quad \rho^* = \rho/\rho_\infty \quad p^* = p/\rho_\infty$$

$$T^* = T/T_\infty \quad e^* = e/V_\infty^2 \;;\; x^* = x/L \quad y^* = y/L \quad z^* = z/L \quad t^* = tV_\infty/L$$

Then defining:

$$\alpha = r^* = (x^{*2} + y^{*2} + z^{*2})^{1/2} \;;\; \beta = \frac{y^*}{x^*} \;;\; \gamma = \frac{z^*}{x^*} \;;\; \lambda = (1+\beta^2+\gamma^2)^{1/2} \;;\; \tau = t^* \; \ldots$$

... we have the following strong conservation form:

$$\frac{\partial}{\partial \tau}\left(\frac{\alpha^2}{\lambda^3} U^*\right) + \frac{\partial}{\partial \alpha}\left(\frac{\alpha^2}{\lambda^4}(F^* + \beta G^* + \gamma H^*)\right) + \frac{\partial}{\partial \beta}\left(\frac{\alpha}{\lambda^2}(-\beta F^* + G^*)\right) + \frac{\partial}{\partial \gamma}\left(\frac{\alpha}{\lambda^2}(-\gamma F^* + H^*)\right) = 0$$

Then assuming $\frac{\partial F^*}{\partial \alpha} = \frac{\partial G^*}{\partial \alpha} = \frac{\partial H^*}{\partial \alpha} = 0$ and computing a solution on a sphere whose nondimensional radius is $r^* = r/L = 1 = \alpha$ we obtain the "conical" Navier-Stokes' Equations:

(24)

$$\boxed{\frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial \beta} + \frac{\partial G}{\partial \gamma} + H = 0}$$

where:

$$U = \frac{U^*}{\lambda^3} \qquad F = \frac{-\beta F^* + G^*}{\lambda^2} \qquad G = \frac{-\gamma F^* + H^*}{\lambda^2}$$

$$H = \frac{2}{\lambda^4}(F^* + \beta G^* + \gamma H^*)$$

In these equations, the partial derivatives appearing in the viscous terms of F^*, G^* and H^* must be transformed using:

$$\frac{\partial}{\partial x^*} = -\beta\lambda\frac{\partial}{\partial \beta} - \gamma\lambda\frac{\partial}{\partial \gamma} \;;\; \frac{\partial}{\partial y^*} = \lambda\frac{\partial}{\partial \beta} \;;\; \frac{\partial}{\partial z^*} = \lambda\frac{\partial}{\partial \gamma}$$

Let us note that the "conical" Navier-Stokes equations depend on the position $r = L$ where they are computed through the Reynolds number $Re_L = \frac{\rho_\infty V_\infty L}{\mu_\infty}$ which appears in the viscous terms. For inviscid solutions, these equations are independent of r and thus truly conical.

The "conical" Navier - Stokes' Equations are a very approximate form of the complete Navier - Stokes Equations and consequently are to be avoided for flow problems where a high degree of accuracy is required. They take a very simple form and can be solved with a time - dependent algorithm.

3.4. The Viscous Shock - Layer Equations.

The Viscous Shock - Layer Equations show many advantages:

– They are simpler than the PNS equations

– Since they remain hyperbolic - parabolic in both the streamwise and crossflow directions, they can be solved by a marching procedure in both directions while the PNS equations which are hyperbolic - parabolic only in the streamwise direction must be solved simultaneously over the entire crossflow plane!

– They are used to solve te viscous flow in blunt nose region where the PNS equations are not applicable since the flow is subsonic in this region. Consequently they provide a starting solution for a subsequent PNS resolution.

However, their major disadvantage is that they cannot be used to compute flowfields with crossflow separation.

The Viscous Shock - Layer Equations are:

(25)

continuity:
$$\frac{\partial}{\partial \xi^*}\left[(r^* + \eta^* \cos\phi)^m \rho^* u^*\right] + \frac{\partial}{\partial \eta^*}\left[(1+K^*\eta^*)(r^*+\eta^*\cos\phi)^m \rho^* v^*\right] = 0$$

ξ momentum:
$$\rho^*\left[\frac{u^*}{1+K^*\eta^*}\frac{\partial u^*}{\partial \xi^*} + v^*\frac{\partial u^*}{\partial \eta^*} + \frac{K^* u^* v^*}{1+K^*\eta^*}\right] + \frac{1}{1+K^*\eta^*}\frac{\partial p^*}{\partial \xi^*}$$
$$= \frac{\epsilon^2}{(1+K^*\eta^*)^2 (r^*+\eta^*\cos\phi)^m}\frac{\partial}{\partial \eta^*}\left[(1+K^*\eta^*)^2 (r^*+\eta^*\cos\phi)^m \tau^*\right]$$

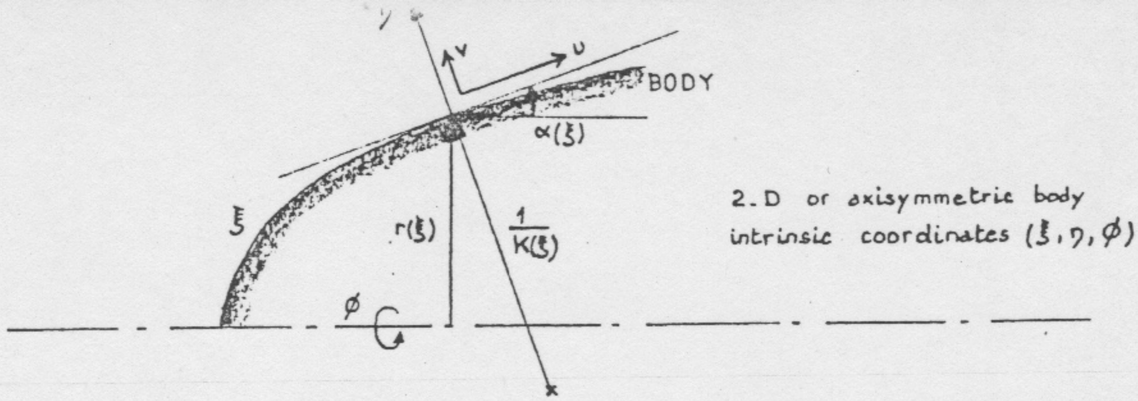
$$\text{where } \tau^* = \mu^*\left(\frac{\partial u^*}{\partial \eta^*} - \frac{K^* u^*}{1+K^*\eta^*}\right)$$

η momentum:
$$\rho^*\left[\frac{u^*}{1+K^*\eta^*}\frac{\partial v^*}{\partial \xi^*} + v^*\frac{\partial v^*}{\partial \eta^*} - \frac{K^*(u^*)^2}{1+K^*\eta^*}\right] + \frac{\partial p^*}{\partial \eta^*} = 0$$

energy:
$$\rho^*\left(\frac{u^*}{1+K^*\eta^*}\frac{\partial T^*}{\partial \xi^*} + v^*\frac{\partial T^*}{\partial \eta^*}\right) - \frac{u^*}{1+K^*\eta^*}\frac{\partial p^*}{\partial \xi^*} - v^*\frac{\partial p^*}{\partial \eta^*} = \frac{\epsilon^2(\tau^*)^2}{\mu^*}$$
$$+ \frac{\epsilon^2}{(1+K^*\eta^*)(r^*+\eta^*\cos\phi)^m}\frac{\partial}{\partial \eta^*}\left[(1+K^*\eta^*)(r^*+\eta^*\cos\phi)^m \frac{\mu^*}{Pr}\frac{\partial T^*}{\partial \eta^*}\right]$$

These equations have been expanded up to second order in ϵ where $\epsilon = \left(\frac{\mu_{ref}}{\rho_\infty V_\infty r_{nose}}\right)^{1/2}$ and μ_{ref} is the coefficient of viscosity evaluated at $T_{ref} = \frac{V_\infty^2}{C_{p_\infty}}$

m = 0 corresponds to a 2-D intrinsic coordinate system and m = 1 to an axisymmetric body intrinsic coordinate system:



The nondimensionalized variables are:

$\xi^* = \xi / r_{nose}$; $\eta^* = \eta / r_{nose}$; $r^* = r / r_{nose}$; $K^* = K / r_{nose}$

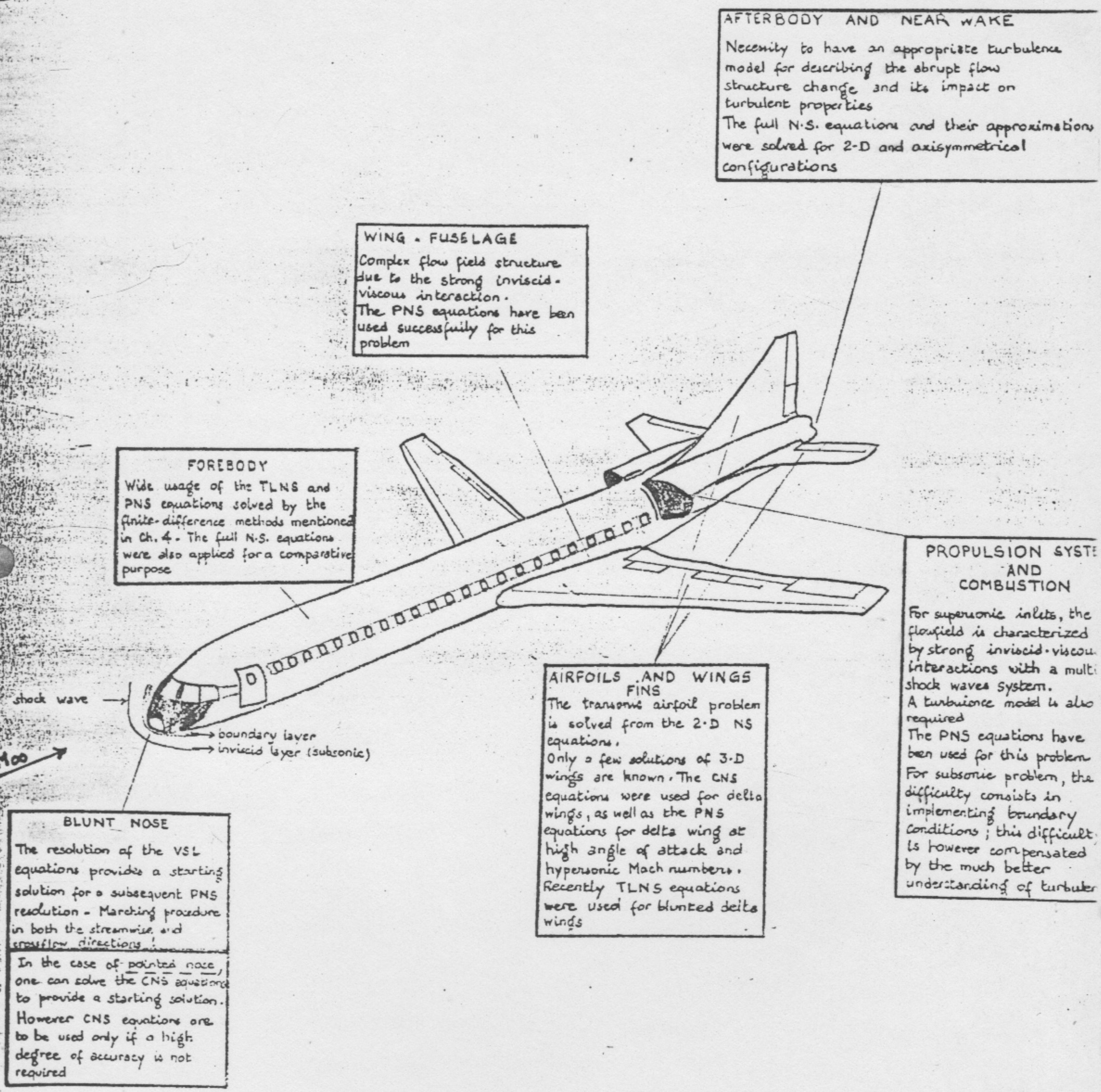
$u^* = u / V_\infty$; $v^* = v / V_\infty$; $\rho^* = \rho / \rho_\infty$; $p^* = p / \rho_\infty V_\infty^2$

$T^* = T / T_{ref}$; $\mu^* = \mu / \mu_{ref}$

In the thin shock layer approximation, the normal (η-) momentum equation becomes

$$\frac{\partial p^*}{\partial \eta^*} = \frac{K^* \rho^* (u^*)^2}{1 + K^* \eta^*}$$

5. Conclusion : summary of the results.



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Newtonian Fluids: Stress is a linear function of the instantaneous velocity gradient.

Non-Newtonian Fluids: Stress is a non-linear function of the hystory of the deformation gradient (fluids with "memory": honey, heavy oils, molten plastic and metals, jellies).

Entropy: Measures the excess amount of heat generated in a process beyond that which can be recovered as work done by adiabatic expansion:

-- slow compression heats the gas, while slow expansion to the original pressure brings it back to the original temperature (all the stored energy is retrieved).

-- fast compression produces compression shock waves that result in excess heating and produce the entropy. Reexpansion to the original pressure leaves the gas at a higher temperature than originally present (not all of the input energy has been retrieved).

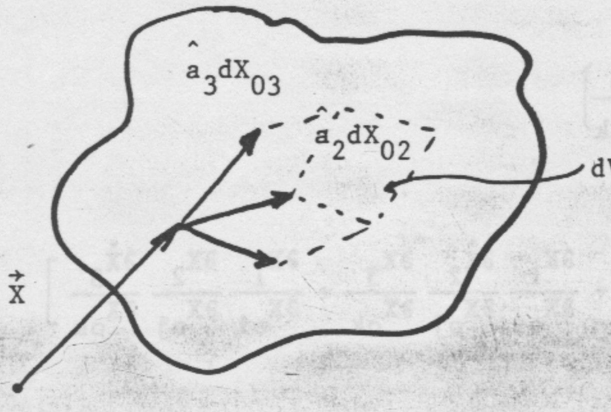
-- a fluid with high velocity fluctuations has more kinetic energy than one with the low velocity fluctuations. Viscosity acts in such a way as to decrease the frequency of velocity fluctuations thus decreasing the kinetic energy which then converts into heat (or internal energy).

-- in the case of a compression shock the conservation laws are satisfied (mass, momentum, energy). If the shock is very thin, then the compression length and time scales are negligibly small. Hence, there will be no velocity fluctuations after the shock, and this would mean that kinetic energy of

the flow decreased, internal energy increased, thus, entropy <u>increased</u> across the compression shock.

<u>NOTE</u>: The system is insulated! Therefore, if $\vec{q} = 0$ (adiabatic walls of the system), the work Φ is not really dissipated and lost, but is retained (as an internal energy) by the system.

Reynolds Transport Theorems (Conservation Laws in Integral Form):



$$\left.\begin{matrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{matrix}\right\}\text{ arbitrary base vectors}$$

$$\hat{a}_i = \frac{\partial \vec{x}}{\partial X_{01}}$$

Then

$$dV = \hat{a}_1 \; dX_{o1} \cdot \left(\hat{a}_2 \; dX_2 \times \hat{a}_3 \; dX_{03}\right)$$

$$dV = \frac{\partial \vec{X}}{\partial X_{01}} \cdot \left(\frac{\partial \vec{X}}{\partial X_{02}} \times \frac{\partial \vec{X}}{\partial X_{03}}\right) dX_{01} \; dX_{02} \; dX_{03}$$

$$dV = \varepsilon_{ijk} \frac{\partial X_i}{\partial X_{01}} \frac{\partial X_j}{\partial X_{02}} \frac{\partial X_k}{\partial X_{03}} dX_{01} \; dX_{02} \; dX_{03}$$

$$dV = \det \begin{vmatrix} \frac{\partial X_1}{\partial X_{01}} & \frac{\partial X_2}{\partial X_{01}} & \frac{\partial X_3}{\partial X_{01}} \\ \frac{\partial X_1}{\partial X_{02}} & \frac{\partial X_2}{\partial X_{02}} & \frac{\partial X_3}{\partial X_{02}} \\ \frac{\partial X_1}{\partial X_{03}} & \frac{\partial X_2}{\partial X_{03}} & \frac{\partial X_3}{\partial X_{03}} \end{vmatrix} dX_{01} \; dX_{02} \; dX_{03}$$

$$\boxed{dV = J \; dV_o}$$

Note also that

$$\frac{dJ}{dt} = \varepsilon_{ijk} \frac{d}{dt} \left(\frac{\partial X_1}{\partial X_{oi}} \frac{\partial X_2}{\partial X_{oj}} \frac{\partial X_3}{\partial X_{ok}} \right)$$

$$= \varepsilon_{ijk} \left[\frac{\partial \dot{X}_1}{\partial X_{oi}} \frac{\partial X_2}{\partial X_{oj}} \frac{\partial X_3}{\partial X_{ok}} + \frac{\partial X_1}{\partial X_{oi}} \frac{\partial \dot{X}_2}{\partial X_{oj}} \frac{\partial X_3}{\partial X_{ok}} + \frac{\partial X_1}{\partial X_{oi}} \frac{\partial X_2}{\partial X_{oj}} \frac{\partial \dot{X}_3}{\partial X_{ok}} \right]$$

$$= \varepsilon_{ijk} \left[\frac{\partial \dot{X}_i}{\partial X_\ell} \frac{\partial X_\ell}{\partial X_{oi}} \frac{\partial X_2}{\partial X_{oj}} \frac{\partial X_3}{\partial X_{ok}} + \text{ 2nd term + 3rd term} \right]$$

But

$$\varepsilon_{ijk} \frac{\partial X_\ell}{\partial X_{oi}} \frac{\partial X_2}{\partial X_{oj}} \frac{\partial X_3}{\partial X_{ok}} = \varepsilon_{\ell 23} J = \begin{matrix} J & \text{if } \ell = 1 \\ 0 & \text{if } \ell = 2,3 \end{matrix}$$

Hence,

$$\frac{dJ}{dt} = \varepsilon_{\ell 23} J \frac{\partial \dot{X}_1}{\partial X_\ell} + \varepsilon_{1\ell 3} J \frac{\partial \dot{X}_2}{\partial X_\ell} + \varepsilon_{12\ell} J \frac{\partial \dot{X}_3}{\partial X_\ell} = J \vec{\nabla} \cdot \dot{\vec{X}}$$

$$\boxed{\frac{dJ}{dt} = J \vec{\nabla} \cdot \vec{u}}$$

Let F be any arbitrary <u>scalar</u> function. Then

$$\frac{d}{dt}\int_V F dV = \frac{d}{dt}\int_{V_o} F\, J dV_o = \int_{V_o} \frac{d}{dt}(FJ)\, dV_o$$

$$= \int_{V_o} \frac{dF}{dt} J dV_o + \int_{V_o} F \frac{dJ}{dt} dV_o$$

$$= \int_V \frac{dF}{dt} dV + \int_{V_o} F \frac{dJ}{dt} dV_o$$

$$= \int_V \frac{dF}{dt} dV + \int_{V_o} F\, J\, \vec{\nabla}\cdot\vec{u}\, dV_o$$

$$= \int_V \frac{dF}{dt} dV + \int_V F\, \vec{\nabla}\cdot\vec{u}\, dV$$

Finally, the First Reynolds Transport Theorem:

$$\frac{d}{dt}\int_V F\, dV = \int_V \left(\frac{dF}{dt} + F\vec{\nabla}\cdot\vec{u}\right)dV$$

Note that it can be further expanded as

$$\frac{d}{dt}\int_V F\, dV = \int_V \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \vec{X}}\frac{\partial \vec{X}}{\partial t} + F\vec{\nabla}\cdot\vec{u}\right)dV$$

$$= \int_V \left[\frac{\partial F}{\partial t} + (\vec{u}\cdot\vec{\nabla}F + F\,\vec{\nabla}\cdot\vec{u})\right]dV$$

Then

$$\frac{d}{dt}\int_V F\, dV = \int_V \left[\frac{\partial F}{\partial t} + \vec{\nabla}\cdot(F\vec{u})\right]dV = \int_V \frac{\partial F}{\partial t} dV + \int_{\partial V} F(\vec{u}\cdot\hat{n})dS$$

CONSERVATION OF MASS (CONTINUITY EQUATION)

$$\int_{V_o} \rho_o dV_o = \int_V \rho dV = \int_{V_o} \rho \; J dV_o$$

Then

$$\int_{V_o} (\rho_o - \rho J) \; dV_o = 0 \quad \therefore \quad \rho_o = \rho J$$

Lagrangian Form of Continuity Equation

Otherwise, use Reynolds First Transport Theorem and, instead of the arbitrary function F, substitute fluid density $\rho = \rho(\vec{X};t)$

Then

$$\frac{d}{dt}\int_V \rho \; dV = \int_V [\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{u}]dV = \int_V [\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})]dV = 0$$

Hence, the two most common forms of the mass conservation equation are:

non-conservative form $\qquad \dfrac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{u} = 0$

conservative (divergence-free) form $\qquad \dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$

Note: If there are any sources or sinks of the mass inside the flow field, the right-hand side of these equations should account for them.

Let $\vec{G}$ be an arbitrary function. Then

$$\frac{d}{dt}\int_V F\vec{G} \; dV = \int_V \frac{d}{dt}[F\vec{G}]dV \quad .$$

Now, use the Reynolds First Transport Theorem and get

$$= \int_V [\frac{d}{dt}(F\vec{G}) + F\vec{G}(\vec{\nabla}\cdot\vec{u})]dV$$

$$= \int_V [\vec{G}\,\frac{dF}{dt} + F\,\frac{d\vec{G}}{dt} + F\vec{G}(\vec{\nabla}\cdot\vec{u})]dV$$

$$= \int_V \{\vec{G}\,[\frac{dF}{dt} + F\,\vec{\nabla}\cdot\vec{u}] + F\,\frac{dG}{dt}\}dV$$

$= 0$ (Reynolds First Transport Theorem)

Then,

$$\boxed{\frac{d}{dt}\int_V F\vec{G}\,dV = \int F\,\frac{d\vec{G}}{dt}\,dV}$$

Reynolds Second Transport Theorem

Say, $F = \rho$ and $\vec{G} = \vec{u}$ where $\vec{u}$ is the fluid velocity vector. Then

$$\frac{d}{dt}\int_V \rho\vec{u}\,dV = \int_V \rho\,\frac{d\vec{u}}{dt}\,dV$$

CONSERVATION OF LINEAR MOMENTUM

Types of forces: i) body forces: $\rho\vec{b}$

ii) surface forces:

pressure (normal)
sheer (tangential)

$\underline{\underline{T}}$ = stress tensor ; $\vec{\tau} = \underline{\underline{T}}\cdot\hat{n}$

$$\frac{d}{dt}\int_V \rho\vec{u}\,dV = \int_V \rho\vec{b}\,dV + \oint_{\partial V} \vec{\tau}\,dS$$

Rate of Change of Linear Momentum	Body Forces (Gravity, Electro-magnetic, Coriolis)	Force of the Traction Vector on V (Pressure and Viscous Forces)

Use the Reynolds Second Theorem on the first term and the divergence theorem on the last term and get

$$\int_V \rho\frac{d\vec{u}}{dt}\,dV = \int_V \rho\vec{b}\,dV + \int_V \vec{\nabla}\cdot\underline{\underline{T}}\,dV$$

or

$$\boxed{\rho\frac{d\vec{u}}{dt} = \rho\vec{b} + \vec{\nabla}\cdot\underline{\underline{T}}}$$

Since

$$\boxed{\underline{\underline{T}} = -p\,\underline{\underline{I}} + \underline{\underline{\sigma}}} \qquad \therefore\ \vec{\nabla}\cdot\underline{\underline{T}} = -\vec{\nabla}p + \vec{\nabla}\cdot\underline{\underline{\sigma}}$$

Hence,

$$\boxed{\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p - \rho\vec{b} - \vec{\nabla}\cdot\underline{\underline{\sigma}} = 0} \qquad \text{Momentum Equation}$$

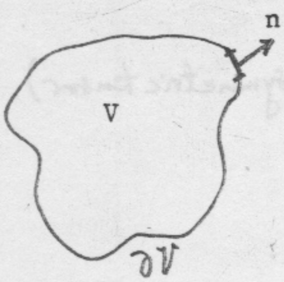
CONSERVATION OF ENERGY

First Law of Thermodynamics: $\Delta e = \delta q - \delta w$

δq = specific heat added to the system

δw = specific work done on the system

New Variables



e = internal energy per unit mass

$\frac{\vec{u}^2}{2}$ = kinetic energy per unit mass

$\dot{q}$ = energy source per unit mass

$\dot{\vec{q}}$ = heat transport vector $(\dot{\vec{q}} = -k\,\vec{\nabla}T)$
Fourier's Law

$\vec{b}$ = body force per unit mass

$$\frac{d}{dt}\int_V \rho\left[\frac{\vec{u}^2}{2} + e\right]dV = \int_V \rho\vec{b}\cdot\vec{u}\,dV + \oint_{\partial V}\vec{\tau}\cdot\vec{u}\,dS + \int_V \rho\dot{q}\,dV - \oint_{\partial V}\dot{\vec{q}}\cdot\hat{n}\,dS$$

Rate of Change of Kinetic plus Internal Energy (Potential Energy Neglected)	Work of Body Forces	Work of Traction Vector	Rate of Heat Generated Internally	Heat Transported Across the Boundary of V into V by Conduction
$\Delta\dot{E}$	$-\delta\dot{W}$		$\delta\dot{Q}$	

To transform this integral equation into a differential equation, first note that

$$\oint_{\partial V}\dot{\vec{q}}\cdot\hat{n}\,dS = \int_V (\vec{\nabla}\cdot\dot{\vec{q}})\,dV$$

$$\oint_{\partial V} \dot{\tau} \cdot \vec{u}\,dS = \oint_{\partial V} (\hat{n}\cdot\underline{\underline{T}}) \cdot \vec{u}\,dS = \oint_{\partial V} n_i\, T_{ij}\, u_j\, dS = \int_V \partial_i (T_{ij}\, u_j) dV$$

$$= \int_V [u_j \partial_i T_{ij} + T_{ij} \partial_i u_j] dV = \int_V [\vec{u} \cdot (\vec{\nabla}\cdot\underline{\underline{T}}) + \underline{\underline{D}} : \underline{\underline{T}}] dV$$

Since $T_{ij}\partial_i u_j = T_{ij} \dfrac{\partial u_j}{\partial x_i} = T_{ij} (D_{ji} + \Omega_{ji}) = T_{ji} D_{ji} = \underline{\underline{D}} : \underline{\underline{T}}$ = stress power

Since $T_{ij}\Omega_{ji} = 0$ ($\underline{\underline{T}}$ is symmetric tensor and $\underline{\underline{\Omega}}$ is an antisymmetric tensor).

Also, use Reynolds' Second Transport theorem to get

$$\frac{d}{dt} \int_V \rho[\frac{\vec{u}^2}{2} + e] dV = \int_V \rho[\vec{u}\frac{d\vec{u}}{dt} + \frac{de}{dt}] dV$$

Finally, balance of energy can be written as (if $\dot{Q} = \rho\dot{q}$)

$$\underbrace{\int_V \vec{u}\cdot [\rho \frac{d\vec{u}}{dt} - \rho\vec{b} - \vec{\nabla}\cdot\underline{\underline{T}}] dV}_{= 0 \text{ (continuity of linear momentum)}} + \int_V \rho \frac{de}{dt} dV = \int_V [\underline{\underline{D}} : \underline{\underline{T}} + \dot{Q} - \vec{\nabla} \cdot \dot{\vec{q}}] dV$$

where $\boxed{\vec{\nabla}\cdot\vec{u} = \mathrm{tr}\underline{\underline{D}}}$

Hence, $\boxed{\rho \dfrac{de}{dt} = \underline{\underline{D}} : \underline{\underline{T}} + \dot{Q} - \vec{\nabla}\cdot\vec{q}}$ Energy Equation

The following variables are assumed to fully characterize the thermodynamic state of a fluid particle.

ρ = density

T = absolute temperature

s = specific entropy

e = specific internal energy

These variables are related by <u>caloric</u> equation of state

$$e = e(\rho,s)$$

$$T = T(\rho,s)$$

The 1st Law of Thermodynamics is given by

$$\int_1^2 \delta q - \int_1^2 \delta w = \int_1^2 de = e_2 - e_1$$

where q = heat transfer to a system (per unit mass)

w = work done by the system (per unit mass)

e = energy of the system = kinetic energy + potential energy + all remaining types of energy (per unit mass)

The symbol δ denotes that the quantity is a path function and, therefore, δ is not a differential.

Gibbs relation is derived by first assuming we have an inviscid fluid undergoing a reversible process. The inviscid fluid assumption then stipulates that all work done by the system can be done only by pressure forces (normal stress). In addition, we neglect the body forces. Thus,

$$\delta w = pdv = p\, d\left(\frac{1}{\rho}\right) \quad \text{where} \quad v = \frac{1}{\rho} = \text{specific volume}$$

Next, the assumption of a reversible process implies that the entropy change is given by:

$$Tds = \delta q$$

Thus, for a reversible process of an inviscid fluid, we have that

$$de_t = Tds - pd\left(\frac{1}{\rho}\right) = de + d\left(\frac{1}{2}u^2\right)$$

where e_t = specific total energy

e = specific internal enegy

u = fluid velocity magnitude.

Since all quantitites are now in terms of exact differentials, we can "divide" by dt to get

$$T \frac{ds}{dt} - p \frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{de_t}{dt}$$

Gibbs relation was derived for an inviscid fluid (with no body forces) undergoing a reversible process. However, the equation contains only state properties of the fluid. That is, there are no path dependent variables. Thus, the equation can be applied to determine the change between two thermodynamic states, regardless of whether the <u>intermediate</u> process is reversible or irreversible.

Therefore, Gibbs relation is valid for all possible changes of state produced by either a reversible or irreversible process as long as the change of state is calculated by integrating Gibbs relation along a reversible path.

Here we assumed that there was no other form of work being added to or removed from the system, such as shaft work, and work done on the fluid by, say, electromagnetic forces. If other types of work are present, Gibbs relation is modified to read

$$T \frac{ds}{dt} - p \frac{d}{dt}\left(\frac{1}{\rho}\right) + \delta w_2 + \delta w_3 + \delta w_4 \ldots = \frac{de_t}{dt}$$

where the δw_i, $2 \le i \le n$ are contributions to the total work due to other processes than pressure forces.

Assume further that each fluid particle satisfies Gibbs relation

$$\boxed{\frac{de}{dt} = T \frac{ds}{dt} - p \frac{d\left(\frac{1}{\rho}\right)}{dt} = \frac{p}{\rho^2} \frac{d\rho}{dt} + T \frac{ds}{dt}}$$

Gibbs relation and the caloric equation of state imply that

$$\boxed{\frac{de}{dt} = \frac{\partial e}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial e}{\partial s}\frac{ds}{dt}} \qquad \therefore \qquad \boxed{\frac{\partial e}{\partial s} = T}$$

$$dT = \frac{\partial T}{\partial \rho}\, d\rho + \frac{\partial T}{\partial s}\, ds \qquad \therefore \qquad \boxed{\frac{\partial e}{\partial \rho} = \left(\frac{p}{\rho^2}\right)}$$

Note that from conservation of mass

$$\frac{d\rho}{dt} + \rho\, \vec{\nabla}\cdot\vec{u} = 0 \qquad \therefore \qquad -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{1}{\rho}\vec{\nabla}\cdot\vec{u} = \frac{1}{\rho}\,\mathrm{tr}\,\underline{\underline{D}}$$

Then

$$\frac{d\left(\frac{1}{\rho}\right)}{dt} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{1}{\rho}\vec{\nabla}\cdot\vec{u} = \frac{1}{\rho}\,\mathrm{tr}\,\underline{\underline{D}} = \frac{1}{\rho}\underline{\underline{D}} : \underline{\underline{I}}$$

Hence, Gibbs relation becomes

$$\frac{de}{dt} = T\frac{ds}{dt} - p\,\frac{d\left(\frac{1}{\rho}\right)}{dt} = T\frac{ds}{dt} - \frac{p}{\rho}\underline{\underline{D}} : \underline{\underline{I}}$$

Combine now Gibbs relation and the balance of energy

$$\rho\frac{de}{dt} = \rho\, T\frac{ds}{dt} - p\,\underline{\underline{D}} : \underline{\underline{I}} = \underline{\underline{D}} : \underline{\underline{T}} + \dot{Q} - \vec{\nabla}\cdot\dot{\vec{q}}$$

Then

$$\boxed{\rho\, T\frac{ds}{dt} = \underline{\underline{D}} : [p\underline{\underline{I}} + \underline{\underline{T}}] + \dot{Q} - \vec{\nabla}\cdot\dot{\vec{q}}}$$

Entropy Generation Equation

VISCOUS DISSIPATION FUNCTION Φ :

$$\Phi = \underline{\underline{D}} : [p\underline{\underline{I}} + \underline{\underline{T}}]$$

Recall that

$$\boxed{\underline{\underline{T}} = \frac{1}{3}\underline{\underline{I}}\,\mathrm{tr}\,\underline{\underline{T}} + \underline{\underline{S}}} \boxed{= -\bar{p}\underline{\underline{I}} + \underline{\underline{S}}} \quad \text{where} \quad \boxed{\mathrm{tr}\,\underline{\underline{S}} = 0}$$

Tensor $\underline{\underline{S}}$ is called the stress deviator or "extra stress" ($\underline{\underline{S}}$ consists of the off-diagonal terms of $\underline{\underline{T}}$).

Note: $p = p(\rho, s)$ = "thermodynamic" pressure

$\bar{p} = -\frac{1}{3}\,\mathrm{tr}\,\underline{\underline{T}}$ = "hydrostatic" or "mechanical" pressure.

For the fluid in motion viscous dissipation function Φ is

$$\boxed{\Phi = \underline{\underline{D}} : [(p-\bar{p})\underline{\underline{I}} + \underline{\underline{S}}]}$$

For fluid at rest: $\boxed{p = \bar{p}}$ $\therefore$ $\boxed{\Phi = \underline{\underline{D}} : \underline{\underline{S}}}$

IMPORTANT: Viscous dissipation function Φ represents conversion of mechanical work to heat.

Hence, entropy production equation can be written as

$$\boxed{\rho T \frac{ds}{dt} = \Phi + \dot{Q} - \vec{\nabla}\cdot\dot{\vec{q}}}$$

Viscosity Heat Source Heat Transfer

Dissipation Principle: The total entropy $\dot{S}_G$ of an adiabatically enclosed system cannot decrease.

$$\boxed{\dot{S}_G \geq 0} \qquad d\dot{S} = \left(\frac{\delta\dot{Q}}{T}\right)_{rev} + d\dot{S}_{irrev}$$

$$\boxed{\int_V \rho \frac{ds}{dt}\, dV = \dot{S}_G - \oint_{\partial V} \frac{\dot{\vec{q}} \cdot \vec{n}}{T}\, dS}$$

$$\boxed{\delta Q = dE + \delta W}$$

First Law of Thermodynamics

Rate at Which System Gains Entropy	Rate at Which Entropy is Generated Internally	Rate at Which Entropy Crosses the System Boundary

Use divergence theorem and dissipation principle ($\dot{S}_G \geq 0$)

$$\dot{S}_G = \int_V [\rho \frac{ds}{dt} + \vec{\nabla}\cdot(\frac{\dot{\vec{q}}}{T})]\, dV \geq 0$$

Since V is arbitrary

$$\rho \frac{ds}{dt} + \vec{\nabla}\cdot(\frac{\dot{\vec{q}}}{T}) = \frac{\Phi + \dot{Q}}{T} - \frac{1}{T}\vec{\nabla}\cdot\dot{\vec{q}} + \vec{\nabla}\cdot(\frac{\dot{\vec{q}}}{T}) \geq 0$$

But

$$-\frac{1}{T}\vec{\nabla}\cdot\vec{q} + \vec{\nabla}\cdot(\frac{\vec{q}}{T}) = -\frac{1}{T}\vec{\nabla}\cdot(k\vec{\nabla}T) + \vec{\nabla}\cdot(\frac{-k\vec{\nabla}T}{T}) = \dot{\vec{q}}\cdot(\frac{1}{T})$$

Hence,

$$\boxed{\frac{\Phi+\dot{Q}}{T} \geq 0 \qquad \vec{q}\cdot\vec{\nabla}(\frac{1}{T}) \geq 0}$$

CLAUSIUS-DUHEIM INEQUALITY
(holds for all processes)

MORE ON DISSIPATION FUNCTION:

Stress: $\boxed{\underline{\underline{T}} = \frac{1}{3}\underline{\underline{I}}\ \mathrm{tr}\ \underline{\underline{T}} + \underline{\underline{S}}}$ (general expression) $= \boxed{-\bar{p}\underline{\underline{I}} + \underline{\underline{S}}}$

Rate of Strain: $\underline{\underline{D}} = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}$;

$$\boxed{\text{tr } \underline{\underline{D}} = D_{11} + D_{22} + D_{33} = \vec{\nabla}\cdot\vec{u}}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stress Deviator: $\boxed{\underline{\underline{S}} = A\underline{\underline{I}} \text{ tr } \underline{\underline{D}} + 2\mu \underline{\underline{D}}}$

Since we know that $\underline{\underline{T}}$ is symmetric $\therefore$ $\boxed{\text{tr } \underline{\underline{S}} = 0}$

Then

$$3A\ \vec{\nabla}\cdot\vec{u} + 2\mu\ \vec{\nabla}\cdot\vec{u} = 0 \quad \therefore \quad (3A + 2\mu)(\vec{\nabla}\cdot\vec{u}) = 0$$

Hence, if $\vec{\nabla}\cdot\vec{u} \neq 0$ (compressible fluid)

$$\boxed{A = -\frac{2}{3}\mu}$$

and we get

$$\boxed{\underline{\underline{S}} = -\frac{2}{3}\mu\ \underline{\underline{I}}(\vec{\nabla}\cdot\vec{u}) + 2\mu\ \underline{\underline{D}}}$$ Stress Deviator or Extra Stress

and

$$\boxed{\underline{\underline{T}} = [-p + (\mu_B - \frac{2}{3}\mu)(\vec{\nabla}\cdot\vec{u})]\underline{\underline{I}} + 2\mu\ \underline{\underline{D}}}$$ Stress Tensor for Compressible Newtonian Fluid in Motion

where $p = p(\rho, s)$ (thermodynamic pressure).

Hence, the difference between thermodynamic pressure, p , and the mechanical (hydraulic) pressure, $\bar{p}$, is

$$p - \bar{p} = \mu_B \, (\vec{\nabla} \cdot \vec{u})$$

μ_B is "bulk" viscosity which, therefore, exists only when the compressible fluid is in motion.

VISCOSITY COEFFICIENTS

μ = shear viscosity

λ = "secondary" or "dilatational" viscosity

μ_B = "bulk viscosity" (sometimes incorrectly called "secondary").

Bulk viscosity is related to the relaxation time (adjusting time) of energy transfer from translational to vibrational modes. Hence, it is more pronounced at higher frequencies.

NOTE: $\lambda = \left(\mu_B - \frac{2}{3}\mu\right)$

-- "Secondary" viscosity never multiplies off-diagonal terms of the strain tensor $\underline{\underline{D}}$.

-- "Shear" viscosity multiplies off-diagonal terms of $\underline{\underline{D}}$.

STOKES HYPOTHESIS

He assumed (and later expressed a little faith in this assumption) that thermodynamic pressure

$$p(\rho, s) = \frac{de}{d(\frac{1}{\rho})}$$

is equal to the mechanical (mean hydraulic) pressure $\bar{p} = \frac{1}{3}\underline{\underline{I}} \text{ tr } \underline{\underline{T}}$, that is, he assumed that

$$\mu_B = \lambda + \frac{2}{3}\mu = 0$$

This is reasonably correct for monoatomic gases (neon, helium, xenon, krypton, argon), but is completely incorrect for most polyatomic gases and especially for bubbly liquids and foams.

Consider incompressible flows when $\vec{\nabla}\cdot\vec{u} = \text{tr}\,\underline{\underline{D}} = 0$. Hence, shear viscosity, μ must be non-negative in order to produce non-negative viscous dissipation function Φ. This implies that $\lambda \geq 0$ (always!). If this is true, then Stokes' hypothesis ($\mu_B = 0$) suggests that

$$\lambda \leq 0 \quad \text{always, that is,} \quad \lambda = -\frac{2}{3}\mu$$

Since μ_B (bulk viscosity) is a measure of a deviation from the Stokes relation

$$\mu_B = \frac{2}{3}\mu + \lambda \ ,$$

It follows that $\mu_B \geq 0$ always if we have a purely dilatational motion (a gaseous sphere oscillating radially!), because, if $\mu_B = 0$ then this sphere of gas would oscillate forever (Φ=0 ?!). Hence, this simple example shows that Stokes was incorrect on this point. Consequently, measured values of secondary viscosity λ turn out to be positive in direct opposition to the Stokes hypothesis. Therefore, remember that for compressible fluids in motion $\lambda > 0$ and $\mu_B \neq 0$

Viscous dissipation function then becomes $\Phi = \underline{\underline{D}} : [(p-\bar{p})\underline{\underline{I}} + \underline{\underline{S}}]$

$$\Phi = \underline{\underline{D}} : [(p + \frac{1}{3}\,\text{tr}\,\underline{\underline{T}})\underline{\underline{I}} + \underline{\underline{S}}] = \underline{\underline{D}} : [p\,\underline{\underline{I}} + \underline{\underline{T}}]$$

$$\boxed{\Phi = (\mu_B - \frac{2}{3}\mu)(\text{tr}\underline{\underline{D}})^2 + 2\mu\,\underline{\underline{D}} : \underline{\underline{D}}} \quad \geq 0$$

Note that $\lambda = \mu_B - \frac{2}{3}\mu$.

Since

$$\underline{\underline{D}} : \underline{\underline{D}} = (D_{11}^2 + D_{22}^2 + D_{33}^2) + 2(D_{12}^2 + D_{23}^2 + D_{13}^2)$$

$$\text{tr}\,\underline{\underline{D}} = D_{11} + D_{22} + D_{33}$$

and since

$$(D_{11}^2 + D_{22}^2 + D_{33}^2) = \frac{1}{3}[(\text{tr}\underline{\underline{D}})^2 + (D_{11}-D_{22})^2 + (D_{22}-D_{33})^2 + (D_{33}-D_{11})^2]$$

we get the most general form for the viscous dissipation function (valid for all kinematically possible motions). Note that <u>all</u> the terms in Φ are <u>positive</u>! Hence, $\Phi \geq 0$ always!

$$\boxed{\mu_B = \frac{2}{3}\mu + \lambda}$$

Consequently, momentum conservation becomes:

$$\rho\frac{d\vec{u}}{dt} = \rho\vec{b} + \vec{\nabla}[-p + (\mu_B - \frac{2}{3}\mu)(\vec{\nabla}\cdot\vec{u})] + 2\vec{\nabla}\cdot(\mu\underline{\underline{D}})$$

or

$$\rho\frac{d\vec{u}}{dt} = \rho\vec{b} + \vec{\nabla}\cdot\underline{\underline{T}}$$

Also

$$\boxed{\begin{aligned}\Phi = \mu_B(\text{tr}\underline{\underline{D}})^2 + \frac{2}{3}\mu[(D_{11}-D_{22})^2 + (D_{22}-D_{33})^2 + (D_{33}-D_{11})^2] \\ + 4\mu[D_{12}^2 + D_{13}^2 + D_{23}^2]\end{aligned}}$$

$$\boxed{\Phi = \lambda(\vec{\nabla}\cdot\vec{u})^2 + 2\mu[\vec{\nabla}\cdot\left((\vec{u}\cdot\vec{\nabla})\vec{u}\right) + \frac{1}{2}(\vec{\nabla}\times\vec{u})^2 - \vec{u}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{u})]}$$

Be aware that the usual values of "the coefficient of viscosity," μ , appearing in the common engineering literature, is actually a combined viscosity: (LONGITUDINAL VISCOSITY)

$$\mu = \frac{4}{3}\mu \quad \left(= 2\mu + \lambda = \mu''\right)$$

This value came from the adopted linear relation between the stress and the strain

$$\tau = -p + \mu \frac{du}{dx}$$

If T_{ij} are the stress tensor components $(i,j = 1,2,3)$ and

$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ are the components of the rate of deformation tensor (where u_i are the velocity components), then

$$\underline{\underline{T}} = \left[-p + (\mu_B - \frac{2}{3}\mu)(\vec{\nabla}\cdot\vec{u})\right]\underline{\underline{I}} + 2\mu\,\underline{\underline{D}}$$

becomes

$$\boxed{T_{ij} = -p\,\delta_{ij} + \lambda D_{ee}\,\delta_{ij} + 2\mu D_{ij}} \qquad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

For one-dimensional flows $u_2 = u_3 = 0$; $u_1 = u$; $T_{11} = T$

so that

$$T = -p + (\lambda + 2\mu)\frac{du}{dx} = -p + \mu\frac{du}{dx} .$$

Hence, $\mu = \lambda + 2\mu$.

If the Stokes relation $(\lambda = -\frac{2}{3}\mu)$ holds, then combined viscosity is $\mu_c = \frac{4}{3}\mu$.

Both shear viscosity (μ) and secondary viscosity (λ) are actually not constants, but are functions of thermodynamic state.

Usually, the assumption that the temperature dependence of combined viscosity (μ_c) is given by a power law

$$\boxed{\mu_c = \mu_o \left(\frac{T}{T_o}\right)^s} \qquad \text{where } \mu_o = \text{const.}$$

is sufficient (although not exact!) for engineering computations.

The exponent s has the value $s = 0.647$ for helium and $s = 0.816$ for argon. These values both lie between the two most frequently used values provided by the kinetic theory of gases, namely, $s = 0.5$ and $s = 1$.

Nevertheless, this simple-minded account of variability of the combined viscosity μ_c is not acceptable if one wants to study shock wave structure where viscous dissipation and heat conduction are extremely important. The combined viscosity coefficient (μ_c) of undissociated air is well approximated by

$$\mu_c = 1.462 \times 10^{-5} \left(\frac{\sqrt{T}}{1 + \frac{112}{T}} \right) \quad \left[\frac{gm}{cm\text{-}sec}\right]$$

where T = absolute temperature in degrees Kelvin.

The well-known formula of Lord Rayleigh suggests the value of exponent s to be $s = 0.75$ and claims its correctness for the temperatures between the freezing and boiling of water

$$\boxed{\frac{\mu_c}{\mu_o} = \left(\frac{T}{T_o}\right)^{0.75}}$$

Yet another formula of the same s-type is

$$\frac{\mu_c}{\mu_o} = \left(\frac{T}{T_0}\right)^{1.5} \left(\frac{T_0 + 223.2}{T + 223.2}\right) \quad \text{with} \quad T[°R]$$

von Mises

(Similar expression is valid for $(\frac{k}{k_0})$)

which is valid in the limits between $T = 32°F$ and $T = 950°F$.

$$\mu_o = 3.73 \times 10^{-7} \text{ [slug/ft.sec]} = 0.000179 \text{ [g/cm.sec]}$$

at $15°C = 288°K = 518.4\ °R$

But, in general, viscosity is the function of temperature, density and pressure.

$$\mu_c = \mu_c(T, \rho, p)$$

Therefore, one should include at least the effect of density on the combined viscosity

$$\mu_c = \frac{4}{3}\mu + \alpha(\rho,T)\frac{\partial u}{\partial x}$$